A Method of Analysis of Symmetrical Four-Port Networks*

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Summary—An analysis of four-arm symmetrical networks such as a branched directional double stub coupler or the hybrid ring (rat race) is presented. The input wave is broken into an even and an odd mode and the vector amplitude out the various arms is computed from the sums or differences of the reflection or transmission coefficients for the two modes. A zero decibel directional coupler is described and its possible use as a duplexer is proposed. The design of multiple stub directional couplers for any degree of coupling is discussed. A method of computing the bandwidth of all these couplers is outlined, and the bandwidth curves, the power out the various arms with respect to frequency of the zero decibel coupler, are computed. A tabulation is made for six different 3 db couplers (even-power split) and their standing wave ratio, evenness of power split and isolation of the fourth arm as a function of frequency assuming perfect performance at the band center.

Introduction

A SYMMETRICAL network is defined as one which has a plane of symmetry as illustrated in Fig. 1. The four arms in the network may be coaxial lines, waveguides, or strip-lines, but this discussion will be limited to coaxial lines, shunt connected.

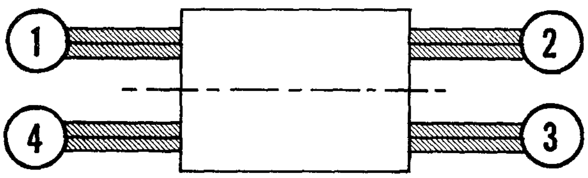

Fig. 1.

It is assumed that the network is lossless, and the ratio of wavelength to line size is very large so that junction effects are negligible.

A signal of unit amplitude is applied at arm 1 and divides in the network. The method of analysis to be described makes possible the determination of the resultant signals appearing at the four arms and how they vary (in phase and amplitude) with frequency.

Symmetry Analysis[1]

If two signals of amplitude $\frac{1}{2}$ and *in* phase are applied at arms 1 and 4, by symmetry a voltage maximum occurs at every point on the line of symmetry. That is, at these points $Z=\infty$ and $Y=0$. This is the equivalent of an open circuit as illustrated in Fig. 2.

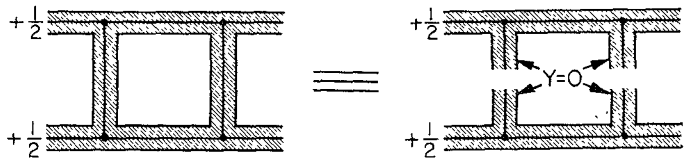

Fig. 2.

Similarly, if two signals of amplitude $\frac{1}{2}$ and *out of* phase are applied at arms 1 and 4, a voltage minimum occurs at every point on the line of symmetry. That is, at these points $Z=0$ and $Y=\infty$. This is the equivalent of a short circuit as in Fig. 3.

Fig. 3.

In each case, the problem reduces to that of a two-arm network. For the even mode, a reflection coefficient $\frac{1}{2}\Gamma_{++}$ and a transmission coefficient $\frac{1}{2}T_{++}$ are determined. Similarly, for the odd mode, $\frac{1}{2}\Gamma_{+-}$ and $\frac{1}{2}T_{+-}$ are determined.

By superposition, the sum of the two cases is a single signal of unit amplitude in arm 1. The resultant signals out of the four arms are also the superposition of the results obtained in the even mode $(++)$ and odd mode $(+-)$ case.

Thus the vector amplitudes of the signals emerging from the four arms are:

$$A_1 = 1/2\Gamma_{++} + 1/2\Gamma_{+-},$$
$$A_2 = 1/2T_{++} + 1/2T_{+-},$$
$$A_3 = 1/2T_{++} - 1/2T_{+-},$$
$$A_4 = 1/2\Gamma_{++} - 1/2\Gamma_{+-}.$$

Matrix Analysis[2]

The analysis of a cascade of two-terminal pair networks may be carried out by use of the *ABCD* matrix. In Fig. 4 the voltages and currents of two-port junctions are related by matrix equations. The matrices of several useful circuit elements are shown in Fig. 5.

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[1] B. A. Lippmann, "Theory of directional couplers," M.I.T. Rad. Lab. Rep., No. 860; December 28, 1945.

[2] W. L. Pritchard, "Quarter-wave coupled waveguide filters," *J. Appl. Phys.*, vol. 18, pp. 862–863; October, 1947.

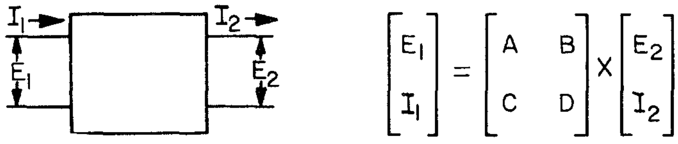


Fig. 4.

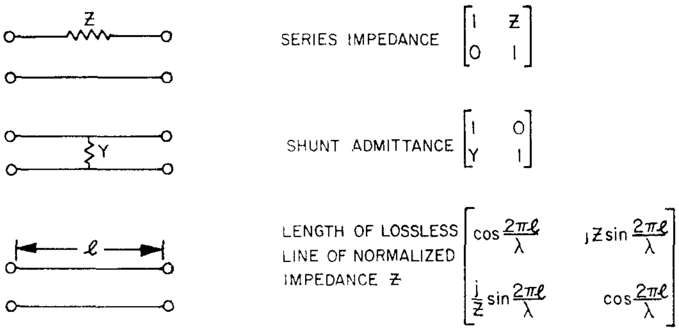


Fig. 5.

The impedance and admittances are all normalized with respect to that of a matched generator and load.

For reciprocity $AD-BC=1$; for forward-to-back symmetry, $A=D$.

$T=\frac{2}{A+B+C+D}=$ insertion voltage transmission coefficient between matched generator and load.

$Z_1=\frac{E_1}{I_1}=\frac{A+B}{C+D}=$ input impedance with matched load.

$\Gamma=\frac{A+B-C-D}{A+B+C+D}=$ voltage reflection coefficient with matched load.

Design at Band Center

The following analysis is for coaxial lines of quarter wavelength long, shunt connected neglecting discontinuity effects. By duality this analysis is valid for waveguide junctions with series connections.

To illustrate an application of the method, consider the 4-arm network illustrated in Fig. 6. Here each line represents a coaxial line, all characteristic admittances are unity, and all lines are a quarter of a wavelength between the pure, shunt junctions.

The matrices for the even and odd modes become:

$$M_{++}=\begin{bmatrix}1&0\\j&1\end{bmatrix}\begin{bmatrix}0&j\\j&0\end{bmatrix}\begin{bmatrix}1&0\\j&1\end{bmatrix}\begin{bmatrix}0&j\\j&0\end{bmatrix}\begin{bmatrix}1&0\\j&1\end{bmatrix}=\begin{bmatrix}0&-j\\-j&0\end{bmatrix},$$

$$M_{+-}=\begin{bmatrix}1&0\\-j&1\end{bmatrix}\begin{bmatrix}0&j\\j&0\end{bmatrix}\begin{bmatrix}1&0\\-j&1\end{bmatrix}\begin{bmatrix}0&j\\j&0\end{bmatrix}\begin{bmatrix}1&0\\-j&1\end{bmatrix}=\begin{bmatrix}0&+j\\+j&0\end{bmatrix}.$$

The even mode matrix is the product of five matrices; the first, third, and last are those of an open-circuited stub one eighth wavelength long; the second and fourth

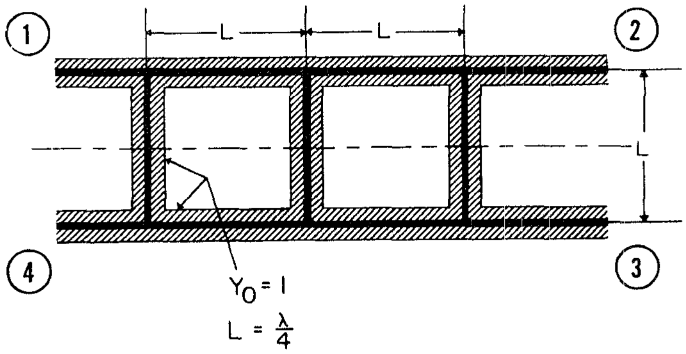


Fig. 6.

are those of a quarter wavelength line. The matrix for the odd mode proceeds similarly except that short circuited stubs are imagined.

$$\Gamma_{++}=\Gamma_{+-}=0 \qquad A_1=A_4=0$$

$$1/2T_{++}=\frac{1}{-2j} \qquad A_2=1/2(T_{++}+T_{+-})=0$$

$$1/2T_{+-}=\frac{1}{2j} \qquad A_3=1/2(T_{++}-T_{+-})=j.$$

Thus $P_1=P_2=P_4=0$ and $P_3=1$ or all the power comes out of arm 3; that is, it is coupled diagonally across the network. The phase is indicated by the value of A_3 which in this case shows a shift of 90°.

A possible application of this device is as a duplexer illustrated in Fig. 7. This is the waveguide analog of the

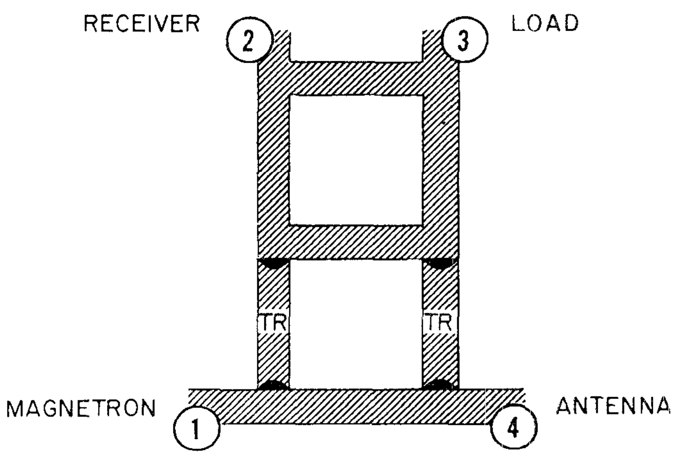


Fig. 7.

circuit of Fig. 6. The magnetron fires the TR tubes and all the power travels to the antenna. Any leakage through the TR's goes diagonally across the network to the load. On reception the tubes look like matched lines and energy travels diagonally across the circuit from the antenna to the receiver.

Another device which can be easily analyzed by this method is the two-stub coupler shown in Fig. 8. (As in Fig. 6, each line in this and all succeeding diagrams represents a coaxial line.)

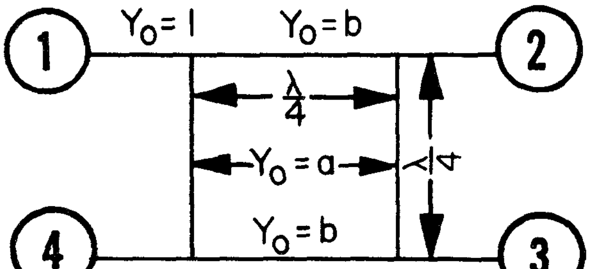


Fig. 8.

Writing the even and odd mode matrices together

$$M_{+\pm} = \begin{bmatrix} 1 & 0 \\ \pm ja & 1 \end{bmatrix} \begin{bmatrix} 0 & \frac{j}{b} \\ jb & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \pm ja & 1 \end{bmatrix} = \begin{bmatrix} \mp \frac{a}{b} & \frac{j}{b} \\ j\left(b - \frac{a^2}{b}\right) & \mp \frac{a}{b} \end{bmatrix}.$$

The device will be matched and perfectly directive if $\Gamma_{++} = \Gamma_{+-} = 0$, that is, if $B = C$ in both the even and odd mode matrices: $1 + a^2 = b^2$. Thus the coupling into arm 3 is

$$20 \log \frac{1}{|A_3|} = 20 \log \frac{\sqrt{b^2 - 1}}{b}.$$

These are the same results as given by Montgomery.[3] For the special case where $a = 1$ and $b = \sqrt{2}$ the device is a 3 db directional coupler for which power in arm 1 divides evenly between arms 2 and 3.

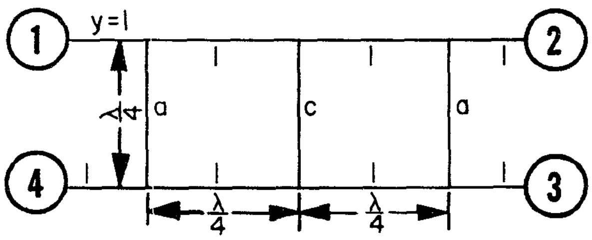


Fig. 9.

If three cross arms are used instead of two (see Fig. 9) it is not necessary to change the admittance of the main lines to achieve match and directivity.

$$M_{+\pm} = \begin{bmatrix} ac - 1 & \mp jc \\ \mp j(2a - a^2c) & ac - 1 \end{bmatrix}.$$

[3] C. G. Montgomery, R. H. Dicke, and E. M. Purcell, "Principles of Microwave Circuits," McGraw-Hill Book Co., Inc., New York, N. Y., pp. 309–310.

For match

$$c = \frac{2a}{1 + a^2} \qquad a = 1 - \frac{\sqrt{1 - c^2}}{c}.$$

Then

$$\text{coupling} = 20 \log_{10} \frac{1}{c} = 20 \log_{10} \frac{1 + a^2}{2a}.$$

Note that for $a = c = 1$ the coupling is 0 db which means that there is no loss in going from arm 1 to arm 3 as noted previously.

A broader band device for three cross arms results when the impedance of the main lines may be changed (see Fig. 10).

$$M_{+\pm} = \begin{bmatrix} \frac{ac}{b^2} - 1 & \mp j \frac{c}{b^2} \\ \mp j\left(2a - \frac{a^2c}{b^2}\right) & \frac{ac}{b^2} - 1 \end{bmatrix}$$

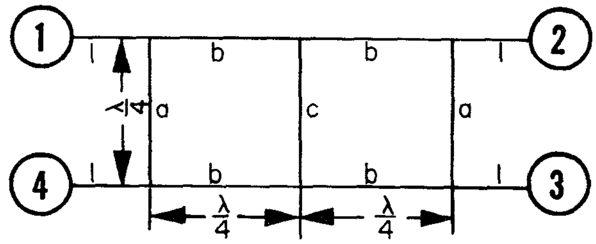


Fig. 10.

For match and perfect directivity

$$B = C \text{ or } c = \frac{2ab^2}{1 + a^2}.$$

Then

$$\text{coupling} = 20 \log_{10} \frac{1}{|A_3|} = 20 \log_{10} \frac{1 + a^2}{2a}.$$

When coupling = 3 db, $a = \sqrt{2} - 1$. A useful form is when $b = c = \sqrt{2}$.

A wider band directional coupler may be made with four cross arms (see Fig. 11).

$$M_{+\pm} = \begin{bmatrix} \pm(a + c - ac^2) & j(c^2 - 1) \\ j(-ac^2 + 2a^2c + a^2 - 1) & \pm(a + c - ac^2) \end{bmatrix}.$$

The values of a and c again can be calculated for perfect match and directivity as before for any coupling.

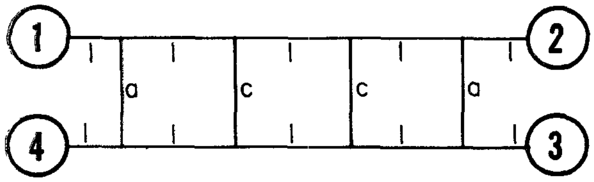


Fig. 11.

With five or six cross arms there could be a third different size for the center arms. However, if the center arms are all kept at the same value, the results for equal power division in arms 2 and 3 are indicated in Table I.

TABLE I

	a	c	
3 arms *aca*	0.4141	0.7071	(Fig. 9)
4 arms *acca*	0.2346	0.5412	(Fig. 11)
5 arms *accca*	0.2088	0.3810	
6 arms *acccca*	0.1464	0.3179	

These values were obtained by setting all the terms in the $M_{+\pm}$ matrix equal in magnitude to $1/\sqrt{2}$. Care was taken that the lowest root of the equation was chosen for broadest band performance.

The rat race ring[4] can also be analyzed (see Fig. 12) and is interesting.

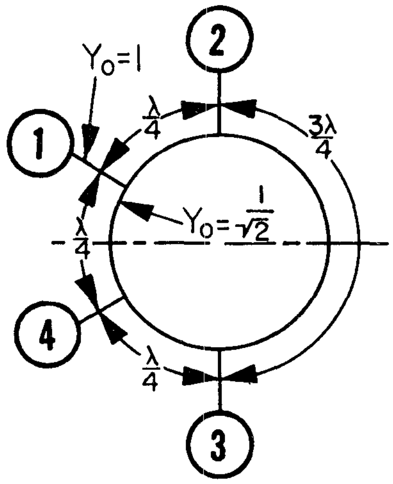


Fig. 12.

$$M_{+\pm} = \begin{bmatrix} 1 & 0 \\ \pm j\frac{1}{\sqrt{2}} & 1 \end{bmatrix} \begin{bmatrix} 0 & j\sqrt{2} \\ \frac{j}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \mp j\frac{1}{\sqrt{2}} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \pm 1 & j\sqrt{2} \\ j\sqrt{2} & \mp 1 \end{bmatrix}.$$

The matrix product is that of an eighth wavelength stub, a quarter wavelength of line and then a three-eighth wavelength stub, all of admittance $1/\sqrt{2}$. Note that in the final matrix $A \neq D$.

As before

$$\Gamma_{++} = \frac{-j}{\sqrt{2}} \qquad A_1 = \frac{\Gamma_{++} + \Gamma_{+-}}{2} = 0,$$

$$\Gamma_{+-} = \frac{j}{\sqrt{2}} \qquad A_4 = \frac{\Gamma_{++} - \Gamma_{+-}}{2} = \frac{-j}{\sqrt{2}},$$

$$T_{++} = \frac{-j}{\sqrt{2}} \qquad A_2 = \frac{T_{++} + T_{+-}}{2} = \frac{-j}{\sqrt{2}},$$

$$T_{+-} = \frac{j}{\sqrt{2}} \qquad A_3 = \frac{T_{++} - T_{+-}}{2} = 0.$$

The result is that power fed in arm 1 divides evenly *in* phase between arm 2 and arm 4 and none is reflected. A similar analysis shows that power fed in arm 2 divides equally and *out of* phase between arms 1 and 3. Similar analysis may be used for the $7/2\lambda$ hybrid ring.[5] (See results, Fig. 15).

Frequency Sensitivity

To calculate the frequency sensitivity of the devices the junctions are assumed still to be pure shunt or series connections. The frequency dependent values, the lengths of line in the matrices are expressed in terms of $t = \tan \pi l/\lambda$ where l is the length of the line. This makes the value of t unity when l is a quarter wavelength. In the case of the three-arm coupler with all elements the same as in Fig. 6 the matrices are as follows:

Even mode stub   Odd mode stub   Length of line

$$\begin{bmatrix} 1 & 0 \\ jt & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ -\frac{j}{t} & 1 \end{bmatrix} \qquad \begin{bmatrix} \frac{1-t^2}{1+t^2} & \frac{j2t}{1+t^2} \\ \frac{j2t}{1+t^2} & \frac{1-t^2}{1+t^2} \end{bmatrix}$$

where $t = \tan \pi l/\lambda (t = 1 \text{ if } l = \lambda/4)$

Note:

$$\cos 2\theta = \frac{1 - \tan^2\theta}{1 + \tan^2\theta} \qquad \sin 2\theta = \frac{2\tan\theta}{1 + \tan^2\theta}.$$

Substituting these in the matrix for the even and odd modes we obtain

$$M_{++} = \frac{1}{(1+t^2)^2} \begin{bmatrix} 1 - 12t^2 + 11t^4 & j(4t - 8t^3) \\ j(7t - 26t^3 + 15t^5) & 1 - 12t^2 + 11t^4 \end{bmatrix},$$

$$M_{+-} = \frac{1}{(1+t^2)^2} \begin{bmatrix} t^4 - 12t^2 + 11 & j(-4t^3 + 8t) \\ j\left(-7t^3 + 26t - \frac{15}{t}\right) & t^4 - 12t^2 + 11 \end{bmatrix}.$$

[4] W. A. Tyrrell, "Hybrid circuits for microwaves," Proc. IRE, vol. 35, pp. 1294–1306; November, 1947.

[5] L. J. Cutrona, "The theory of biconjugate networks," Proc. IRE, vol. 39, pp. 827–832; July, 1951.

Note that for $t=1$ these expressions reduce to those following Fig. 6. For any frequency a value of t can be found and the matrices evaluated and then the vector amplitudes of the waves out all arms calculated.

On the graph in Fig. 13 (below) are plotted the powers out the various arms expressed in decibels below incident power as a function of t and, assuming coaxial lines, also as a function of the ratio of frequency to design frequency. Note the change in scale for the power out arm 3.

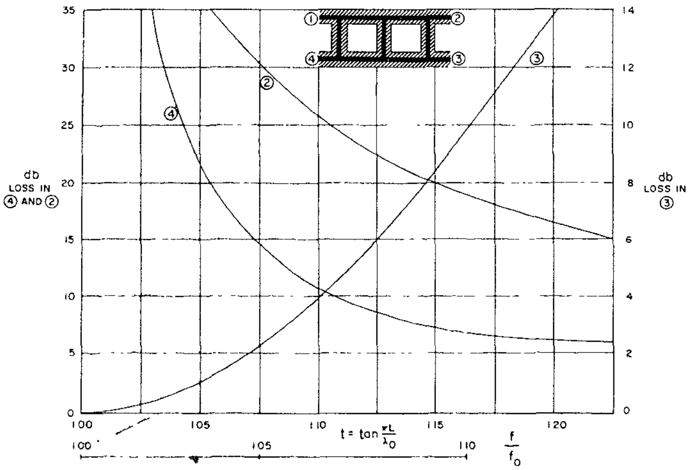


Fig. 13.

The values on these curves for a given value of t are exactly equal to those for $1/t$. Thus for a design over a band of frequencies the arithmetic average frequency should be used for the value of length to get equal performance at the edges of the band.

In the calculations of these curves and of the others that follow, a simplification can be made which makes it unnecessary to multiply out the M_{+-} matrix as a function of t. To obtain the numerical values of the M_{+-} matrix for a given t, the value $1/t$ is put in for t in the M_{++} matrix. Then if the exponent of the term $(1+t^2)^n$ is odd, the sign of the A and D components are changed; if even, those of the B and C components are changed.

In Fig. 14 are shown the theoretical preformance curves calculated by this method for the two simplest four-arm junctions, the rat race, and the double stub 3-db coupler. Both of these can be considered as 3 db directional couplers and will divide power equally between two arms shown with arrows and no power will go out the fourth arm with no arrow, as was shown previously.

On the graph the curves sloping *up* to the right indicated the input vswr with matched loads on the other three arms. Also the curves sloping *down* to the right show the ratio of power coupled to the fourth arm (arm 3 in the case of the rat race) to the incident power. These curves are plotted as a function of t and also as a function of f/f_0 assuming free space variation of wavelength as before. The rat race is a solid line and the square hybrid a dashed line.

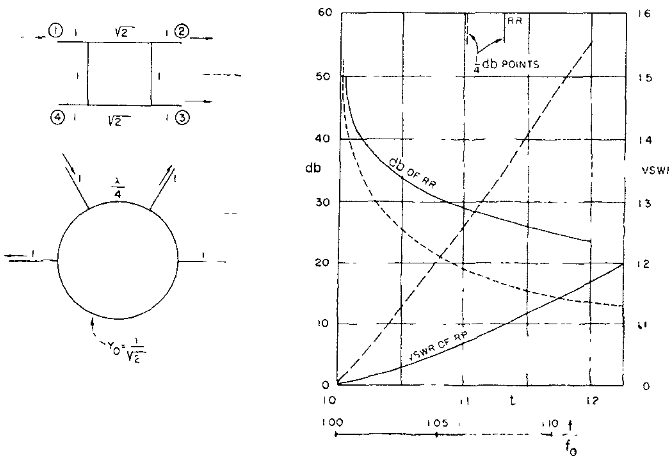


Fig. 14.

While the power will divide evenly between the two arms with arrows when $t=1$, that is, when all values are a quarter wavelength, it will not divide evenly at other frequencies. The values of t at which the ratio goes up to a quarter of a decibel are shown as the little vertical lines at the top of the graph.

The M_{++} matrices are given below for reference for these 3-db couplers. The M_{+-} matrices are not needed if the rule mentioned before is used.

Square hybrid of Fig. 8:

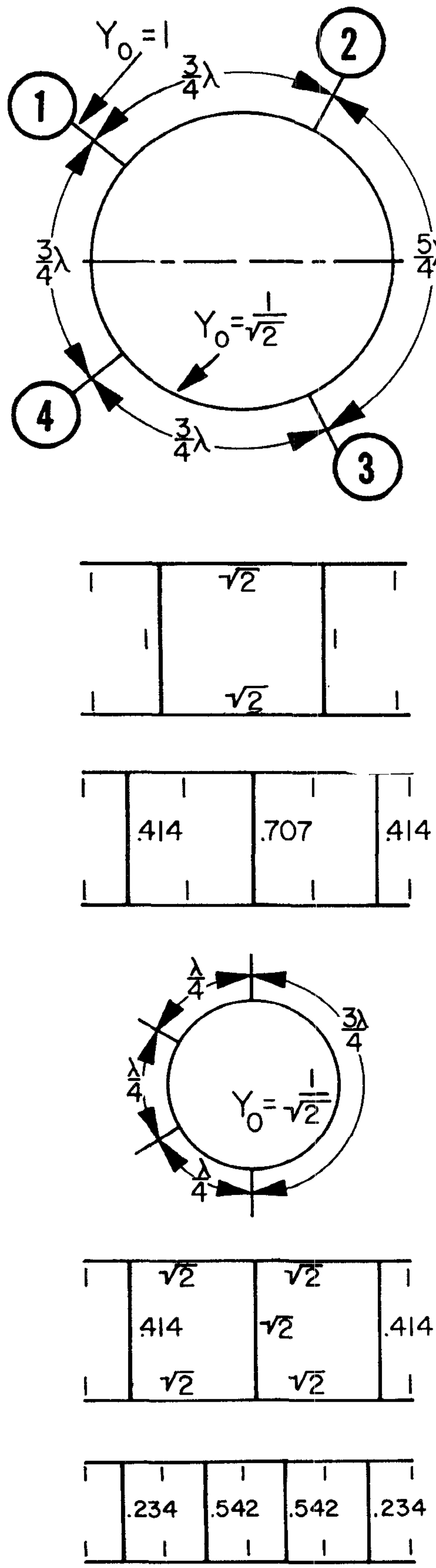
$$M_{++}=\frac{1}{1+t^2}\begin{bmatrix} 1-(1+\sqrt{2})t^2\cdot & j\sqrt{2}t \\ j[(2+2\sqrt{2})t-(2+\sqrt{2})t^3] & 1-(1+\sqrt{2})t^2 \end{bmatrix}$$

Rat race of Fig. 12:

$$M_{++}=\frac{1}{1+t^2}\begin{bmatrix} \dfrac{1-10t^2+5t^4}{1-3t^2} & 2j\sqrt{2}t \\ j(3\sqrt{2}t-\sqrt{2}t^3) & 1-3t^2 \end{bmatrix}.$$

A tabulation of calculated results for six different 3-db couplers described in the text appears in Fig. 15 in order of increasing bandwidth. For $t=1.1$ and 1.2 the first column shows the input vswr, the second shows the ratio of power delivered to arm 4 (arm 3 in the hybrid rings cases) to that incident and the third column shows the ratio of power delivered to arms 2 and 3, (2 and 4 for the hybrid rings cases).

The method outlined in this paper has been the subject of many experimental checks. We have built a coaxial line model of Fig. 7, the duplexer, in $\frac{3}{8}$-inch line at L band. The data taken on this device checked quite consistently with the theory. Also a square hybrid (Fig. 8) has been constructed at L band in $\frac{3}{8}$ inch coaxial



t = 1.10 ($\frac{f}{f_0}$ = 1.06)			t = 1.20 ($\frac{f}{f_0}$ = 1.13)		
VSWR	DB	DIFF IN DB	VSWR	DB	DIFF IN DB
1.37	17.7	1.47			
1.26	19.0	.24	1.57	13.8	.74
1.08	27.4	.18	1.20	20.5	.60
1.07	29.3	.14	1.17	23.3	.51
1.03	37.0	.12	1.12	25.3	.49
1.01	45.0	.10	1.05	32.0	.45

Fig. 15—Tabulation of theoretical results on 3-db directional couplers.

line and its performance checked the curves presented almost exactly. The curves presented on the rat race check very well with published data.[6]

Again it must be emphasized that pure shunt junctions are assumed neglecting fringing effects. This will only be true if the ratio of wavelength to line size is very high. This first approximation is valuable anyway. To make a coaxial 3-db directional coupler with many arms as shown in Fig. 11 requires excessively small center conductors since the characteristic admittance varies as the logarithm of the size ratio. But with waveguide a branch guide 3-db coupler, with many arms appears to be a definite possibility since the impedance of the arms varies as the height of the guide and the fringing effects get smaller as the height of the guide is decreased.

[6] H. T. Budenbom, "Some quasi-biconjugate networks and related topics," Proc. of the Symposium on Modern Network Synthesis, Polytech. Inst. of Brooklyn, New York, N. Y., 1952, pp. 312–326.

Broad-Band Waveguide Series *T* for Switching*

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Summary—**By use of properly proportioned half-wavelength transformer sections in the arms of a waveguide series *T* broad-band performance can be obtained for switching applications. Over the frequency band 8200 to 9765 megacycles per second, corresponding to a bandwidth of 17.4 per cent, an experimental model showed an insertion vswr of less than 1.15 for transmission through the aligned arms and 1.30 for transmission around the bend. Further bandwidth improvement is possible with the use of a special arrangement of quarter-wave transformer sections but at the expense of further impairment of power-carrying capacity.**

Introduction

ONE APPLICATION of a *T* junction is duplexing. The transmitter and antenna are usually connected to the aligned arms and the receiver to the side arm. Power flow from transmitter to antenna and not to the receiver can be considered to be so directed by an appropriately positioned effective short in the side arm. The received signal can be considered to be directed from antenna to side arm by an appropriately positioned effective short in the transmitter arm. An inherent bandwidth limitation for the ordinary *T* is that the effective short positions are correct only for the center frequency and thus give rise to "branching loss,"[1] principally through reflection of energy back out through the antenna. The device discussed in this paper provides one means for minimizing this branching loss over a broader range of frequencies.

In any practical case of duplexer design many special problems relating to positioning of the shorts and special means of solving them are introduced.[1] This paper is directed chiefly at the frequency sensitivity of the stubbed *T* structure itself and means for overcoming it. Fixed metallic shorts are assumed to replace the effective shorts indicated above. For this reason the device is called a switching *T* rather than duplexer.

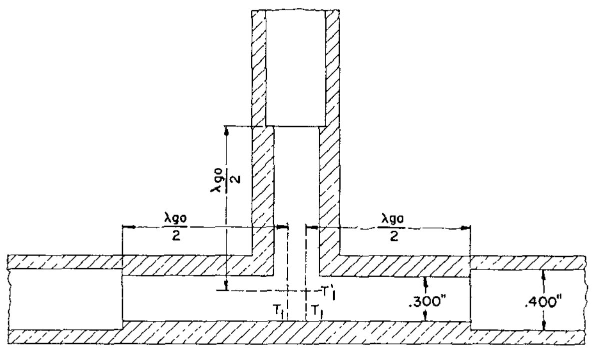

Fig. 1—Waveguide switching *T* cross section—side view.

Shown in Fig. 1 is a dimensioned sketch of the broad-banded series (*E* plane) *T* in the *X* band, RG-52/U rectangular waveguide designed for a center frequency of 9000 mc and a theoretical bandwidth of 16 per cent for a maximum vswr of 1.08. This design was based on circuital computation of a pure series arrangement of the arms and use of the equivalent circuit[2] for the *E* plane *T* at 9000 megacycles per second. At appropriate reference planes the particular equivalent circuit chosen is a good approximation to a series con-

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[1] L. Smullin and C. Montgomery, "Microwave Duplexers," Rad. Lab. Ser., McGraw-Hill Book Co., Inc., New York, N. Y., vol. 14, ch. 7, 1948.

[2] N. Marcuvitz, "Waveguide Handbook," Rad. Lab. Ser., McGraw-Hill Book Co., Inc., vol. 10, pp. 337–351, 1951.